Classification of discrete equations linearizable by point transformation on a square lattice

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Abstract

We provide a complete set of linearizability conditions for nonlinear partial difference equations defined on four points and, using them, we classify all linearizable multilinear partial difference equations defined on four points up to a Möbious transformation.

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1 Introduction

In a series of papers [1–4] one has provided necessary conditions for the linearizability of real dispersive multilinear difference equations on a quad–graph (see Fig. 1). and on three points (Fig. 2).

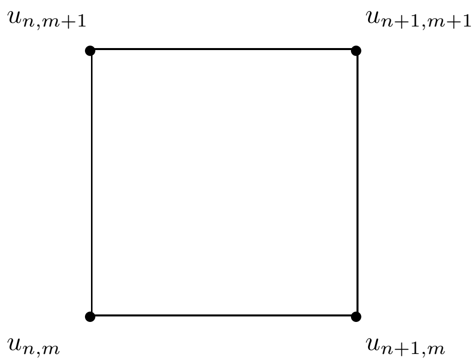

Figure 1: The quad–graph where a partial difference equation is defined

These conditions, obtained by considering the existence of point transformations and symmetries have been sufficient to classify the multilinear equations defined on three points [2] but not those defined on four points. In [4] we considered the problem from the point of view of the symmetries, both point and nonlocal. In this way we get a different set of conditions with respect to those obtained before which, however, are not yet sufficient to classify the multilinear equations defined on four points. So here, using the experience of [2] and [4] we construct the largest possible set of linearizability conditions and, through them, we classify the multilinear equations on a square lattice. We assume a partial difference equation on a quad–graph to be given by

$$\mathcal{E} = \mathcal{E}\left(u_{n,m}, u_{n,m+1}, u_{n+1,m}, u_{n+1,m+1}\right) = 0, \quad \frac{\partial \mathcal{E}}{\partial u_{n+i,m+j}} \neq 0,\ i,j = 0,1, \tag{1}$$

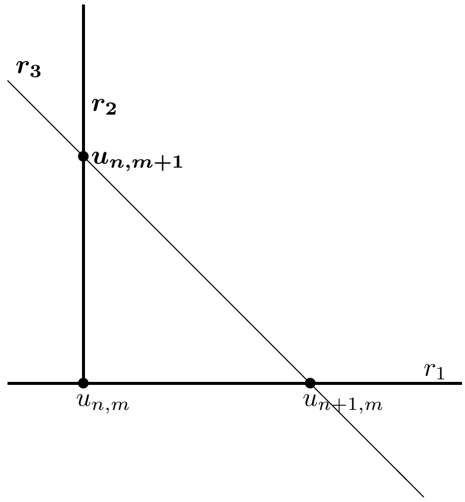

Figure 2: Points related by an equation defined on three points.

for a field $u_{n,m}$ which linearizes into a linear autonomous equation for $\widetilde{u}_{n,m}$

$$a\widetilde{u}_{n,m} + b\widetilde{u}_{n+1,m} + c\widetilde{u}_{n,m+1} + d\widetilde{u}_{n+1,m+1} + e = 0 \tag{2}$$

with a, b, c, d and e being (n, m)–independent arbitrary non zero complex coefficients. The choice that (2) be autonomous is a restriction but it is also a natural simplifying ansatz when one is dealing with autonomous equations. Moreover, as (1, 2) are taken to be autonomous equations, i.e. they have no n, m dependent coefficients, they are translationally invariant under shifts in n and m. So we can with no loss of generality choose as reference point $n = 0$ and $m = 0$. This will also be assumed to be true for the linearizing point tranformation.

By a *linearizing point transformation* we mean a transformation

$$\widetilde{u}_{0,0} = f(u_{0,0}) \tag{3}$$

between (2) and (1) characterized by a function depending just from the function $u_{0,0}$ and on some constant parameters. It will be a *Lie point transformation* if $f = f_{0,0}$ satisfies all Lie group axioms and, in particular, the composition law. In the following we will require much less, i.e. we will only assume the differentiability of the function f up to at least second order.

In Section 2 we discuss point transformations, present the linearizability conditions which ensure that the given equation is linearizable and the differential equations which define the transformation f. In Section 3 we classify all multilinear equations which belong to the class (1) up to a Möbious transformation while in the final Section we present some conclusive remarks and open problems.

2 Discrete equations defined on a square linearizable by a point transformation.

In the autonomous case a generic partial difference equation (1) for the complex function $u_{n,m} \doteq u_{0,0}$ can be rewritten as

$$\mathcal{E}\left(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}\right) = 0, \quad \frac{\partial \mathcal{E}}{\partial u_{i,j}} \neq 0,\ i, j = 0, 1, \tag{4}$$

We will assume that we can solve (4) with respect to each one of the four variables in its argument

$$u_{1,1} = F\left(u_{0,0}, u_{1,0}, u_{0,1}\right), \quad F_{,u_{0,0}} \neq 0 \quad F_{,u_{1,0}} \neq 0, \quad F_{,u_{0,1}} \neq 0, \tag{5a}$$
$$u_{1,0} = G\left(u_{0,0}, u_{0,1}, u_{1,1}\right), \quad G_{,u_{0,0}} \neq 0, \quad G_{,u_{0,1}} \neq 0, \quad G_{,u_{1,1}} \neq 0, \tag{5b}$$
$$u_{0,1} = S\left(u_{0,0}, u_{1,0}, u_{1,1}\right), \quad S_{,u_{0,0}} \neq 0, \quad S_{,u_{1,0}} \neq 0 \quad S_{,u_{1,1}} \neq 0, \tag{5c}$$
$$u_{0,0} = T\left(u_{1,0}, u_{0,1}, u_{1,1}\right), \quad T_{,u_{1,0}} \neq 0, \quad T_{,u_{0,1}} \neq 0 \quad T_{,u_{1,1}} \neq 0, \tag{5d}$$

and that (4) can be linearized by the linearizing autonomous point transformation (3) into the linear equation (2) for the complex function $u_{n,m} \doteq u_{0,0}$, i.e

$$a\widetilde{u}_{0,0} + b\widetilde{u}_{1,0} + c\widetilde{u}_{0,1} + d\widetilde{u}_{1,1} + e = 0. \tag{6}$$

Hence, assuming we can solve (4) with respect to $u_{1,1}$, we can choose as independent variables $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$ and we will have that

$$af_{0,0} + bf_{1,0} + cf_{0,1} + df_{1,1}|_{u_{1,1}=F} + e = 0, \tag{7}$$

must be identically satisfied for any $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$. Differentiating (7) with respect to $u_{0,0}$, $u_{1,0}$ or $u_{0,1}$, we obtain

$$a\frac{df_{0,0}}{du_{0,0}} + d\frac{df_{1,1}}{du_{1,1}}|_{u_{1,1}=F}F_{,u_{0,0}} = 0, \tag{8a}$$

$$b\frac{df_{1,0}}{du_{1,0}} + d\frac{df_{1,1}}{du_{1,1}}|_{u_{1,1}=F}F_{,u_{1,0}} = 0, \tag{8b}$$

$$c\frac{df_{0,1}}{du_{0,1}} + d\frac{df_{1,1}}{du_{1,1}}|_{u_{1,1}=F}F_{,u_{0,1}} = 0, \tag{8c}$$

which have to be identically satisfied for any $u_{0,0}$, $u_{1,0}$ and $u_{0,1}$. From them, considering that $\frac{df(x)}{dx} \neq 0$, we derive that $d \neq 0$, otherwise $a = b = c = e = 0$. As a consequence, considering that also $F_{,u_{0,0}} \neq 0$, $F_{,u_{1,0}} \neq 0$ and $F_{,u_{0,1}} \neq 0$, we have $a \neq 0$, $b \neq 0$ and $c \neq 0$. Then in all generality we can divide (6) by d and, introducing the new parameters $\alpha \doteq a/d \neq 0$, $\beta \doteq b/d \neq 0$, $\gamma \doteq c/d \neq 0$ and $\epsilon \doteq e/a$, (6) can be rewritten as

$$\alpha\widetilde{u}_{0,0} + \beta\widetilde{u}_{1,0} + \gamma\widetilde{u}_{0,1} + \widetilde{u}_{1,1} + \epsilon = 0. \tag{9}$$

Defining $\frac{df(x)}{dx} \doteq H(x)$, from (8) we obtain

$$\frac{F_{,u_{0,0}}}{F_{,u_{1,0}}} = \frac{\alpha H(u_{0,0})}{\beta H(u_{1,0})}, \tag{10a}$$

$$\frac{F_{,u_{0,0}}}{F_{,u_{0,1}}} = \frac{\alpha H(u_{0,0})}{\gamma H(u_{0,1})}. \tag{10b}$$

From (10) we get the following linearizability conditions

$$A(x, u_{0,1}) \doteq \frac{F_{,u_{0,0}}}{F_{,u_{1,0}}}|_{u_{0,0}=u_{1,0}=x} = \frac{\alpha}{\beta}, \quad \forall x,\ u_{0,1}, \tag{11a}$$

$$B(x, u_{1,0}) \doteq \frac{F_{,u_{0,0}}}{F_{,u_{0,1}}}|_{u_{0,0}=u_{0,1}=x} = \frac{\alpha}{\gamma}, \quad \forall x,\ u_{1,0}, \tag{11b}$$

$$C(x, u_{0,0}) \doteq \frac{F_{,u_{0,1}}}{F_{,u_{1,0}}}|_{u_{1,0}=u_{0,1}=x} = \frac{\gamma}{\beta}, \quad \forall x,\ u_{0,0}, \tag{11c}$$

$$\frac{\partial}{\partial u_{0,1}}\frac{F_{,u_{0,0}}}{F_{,u_{1,0}}} = 0, \quad \forall u_{0,0},\ u_{1,0},\ u_{0,1}, \tag{11d}$$

$$\frac{\partial}{\partial u_{1,0}}\frac{F_{,u_{0,0}}}{F_{,u_{0,1}}} = 0, \quad \forall u_{0,0},\ u_{1,0},\ u_{0,1}, \tag{11e}$$

$$\frac{\partial}{\partial u_{0,0}}\frac{F_{,u_{0,1}}}{F_{,u_{1,0}}} = 0, \quad \forall u_{0,0},\ u_{1,0},\ u_{0,1}. \tag{11f}$$

Alternatively the conditions (11a-11c) can be substituted by the following ones

$$\frac{d}{dx}A\left(x,u_{0,1}\right)=0,\quad \forall x,\ u_{0,1}, \tag{12a}$$

$$\frac{d}{dx}B\left(x,u_{1,0}\right)=0,\quad \forall x,\ u_{1,0}, \tag{12b}$$

$$\frac{d}{dx}C\left(x,u_{0,0}\right)=0,\quad \forall x,\ u_{0,0}. \tag{12c}$$

Taking the (principal value of the) logarithm of (8a), we have

$$\log\frac{df_{0,0}}{du_{0,0}}-\log\frac{df_{1,1}}{du_{1,1}}|_{u_{1,1}=F}=\log\left(-\frac{F_{,u_{0,0}}}{\alpha}\right)\ (\mathrm{mod}\,2\pi\mathrm{i})\,. \tag{13}$$

Then, let us introduce the linear operator $\mathcal{B}$

$$\mathcal{B}\doteq\frac{\partial}{\partial u_{0,0}}-\frac{F_{,u_{0,0}}}{F_{,u_{1,0}}}\frac{\partial}{\partial u_{1,0}}, \tag{14}$$

such that and $B\phi\left(F\left(u_{0,0},u_{1,0},u_{0,1}\right)\right)=0$, where ϕ is an arbitrary functions of its argument. When we apply (14) to (13), we obtain an ordinary differential equation describing the evolution of the linearizing transformation

$$\frac{d}{du_{0,0}}\log\frac{df_{0,0}}{du_{0,0}}=\frac{1}{F_{,u_{0,0}}F_{,u_{1,0}}}W_{(u_{0,0})}\left[F_{,u_{1,0}};F_{,u_{0,0}}\right], \tag{15}$$

where $W_{(x)}\left[f;g\right]\doteq fg_{,x}-gf_{,x}$ stands for the Wronskian of the functions f and g. Let's remark that the linearizability conditions (11d-11f) imply that the right hand member of the equation (15) does not depend on $u_{1,0}$ and $u_{0,1}$. These conditions were considered in [3] and had not been sufficient to classify (1). Other similar conditions can be obtained starting from (8b) or (8c). The linearizability conditions presented here have been obtained starting from (5a). Similar results could be obtained starting from (5b), (5c) or (5d). However these results would not have provided any really new linearizability condition. So, here, in the next Section we start the classifying process from the more basic equations (11) as we did in the case of equations depending on just three points [4].

3 Classification of complex autonomous multilinear partial difference equations defined on four points linearizable by a point transformation.

Let $\mathcal{E}\left(u_{0,0},u_{1,0},u_{0,1},u_{1,1}\right)=0$ be the complex multilinear equation

$$\begin{aligned} a_1u_{0,0} &+ a_2u_{1,0}+a_3u_{0,1}+a_4u_{1,1}+a_5u_{0,0}u_{1,0}+a_6u_{0,0}u_{0,1}+ \\ &+ a_7u_{0,0}u_{1,1}+a_8u_{1,0}u_{0,1}+a_9u_{1,0}u_{1,1}+a_{10}u_{0,1}u_{1,1}+ \\ &+ a_{11}u_{0,0}u_{1,0}u_{0,1}+a_{12}u_{0,0}u_{1,0}u_{1,1}+a_{13}u_{0,0}u_{0,1}u_{1,1}+ \\ &+ a_{14}u_{1,0}u_{0,1}u_{1,1}+a_{15}u_{0,0}u_{1,0}u_{0,1}u_{1,1}+a_{16}=0, \end{aligned} \tag{16}$$

where a_j, $j=1,\ldots$, 16, are arbitrary complex free parameters. This equation is invariant under a Möbious transformation of the dependent variable

$$u_{0,0}\doteq\frac{b_1v_{0,0}+b_2}{b_3v_{0,0}+b_4}, \tag{17}$$

where b_k, $k=1,\ldots,4$ are four arbitrary complex parameters such that $b_1b_4-b_2b_3\neq 0$. As we operate in the field of complex numbers and we classify up to Möbious transformations, using inversions, dilations and

translations we can always simplify (16) by setting either **1)** $a_{15} = a_{16} = 0$ or **2)** $\sum_{k=1}^{4} a_k = \sum_{k=5}^{10} a_k = \sum_{k=11}^{14} a_k = a_{15} = 0$, $a_{16} = 1$. Let's now apply to these two cases the six necessary linearizability conditions (11). This amounts to solving a system of 96 algebraic in general nonlinear equations involving the coefficients a_j, $j = 1, \ldots, 14$. Their solution implies, through the integration of the differential equation (15), that the function $f(x)$ appearing in the linearizing transformation (2) can only be of the following two types:

1. The fractional linear function

$$f(x) = \frac{c_1 x + c_2}{c_3 x + c_4}, \quad c_1 c_4 - c_2 c_3 \neq 0, \tag{18}$$

which, as the classification is up to Möbious transformations of $u_{0,0}$, can always be reduced to be $f(x) = 1/x$;

2. The (principal branch) of the logarithmic function

$$f(x) = d_1 \log\left(\frac{d_2 x + d_3}{d_4 x + d_5}\right) + d_6 \quad d_1 \neq 0, \quad d_2 d_5 - d_3 d_4 \neq 0, \tag{19}$$

which, as the classification is up to Möbious transformations, can always be reduced to $f(x) = \log(x)$. Moreover in this case the ratios α/β and α/γ are always real and of modulus 1, so that the possible linear equations can be only of the following four types:

$$\alpha\left(\widetilde{u}_{0,0} + \widetilde{u}_{1,0} + \widetilde{u}_{0,1}\right) + \widetilde{u}_{1,1} + \epsilon = 0; \tag{20a}$$
$$\alpha\left(\widetilde{u}_{0,0} + \widetilde{u}_{1,0} - \widetilde{u}_{0,1}\right) + \widetilde{u}_{1,1} + \epsilon = 0; \tag{20b}$$
$$\alpha\left(\widetilde{u}_{0,0} - \widetilde{u}_{1,0} + \widetilde{u}_{0,1}\right) + \widetilde{u}_{1,1} + \epsilon = 0; \tag{20c}$$
$$\alpha\left(\widetilde{u}_{0,0} - \widetilde{u}_{1,0} - \widetilde{u}_{0,1}\right) + \widetilde{u}_{1,1} + \epsilon = 0. \tag{20d}$$

It is easy to prove that, if the transformation $\widetilde{u}_{0,0} \doteq \log(u_{0,0})$ has to produce a multilinear equation for $u_{0,0}$, we must have $\alpha = \pm 1$. In fact, as $F(u_{0,0}, u_{1,0}, u_{0,1})$ should be a fractional linear function of $u_{0,0}$ with coefficients depending on $u_{1,0}$ and $u_{0,1}$, we have that the relations

$$u_{0,0}^{\alpha}\left(e_1 u_{0,0} + e_2\right) = e_3 u_{0,0} + e_4, \quad e_j = e_j\left(u_{1,0}, u_{0,1}\right), \quad j = 1, \ldots, 4,$$

where $e_1 e_4 - e_2 e_3$ is not identically zero for all $u_{1,0}$ and $u_{0,1}$, must be identically satisfied for all $u_{0,0}$. Differentiating (21) twice with respect to $u_{0,0}$, we get $e_1(\alpha + 1) u_{0,0} + e_2(\alpha - 1) = 0$ identically for all $u_{0,0}$, so that

$$e_1(\alpha + 1) = 0, \quad e_2(\alpha - 1) = 0.$$

Considering that e_1 and e_2 cannot be simultaneously identically zero, it follows that $\alpha = \pm 1$. In this way we obtain a set of eight linear equations corresponding to eight linearizable nonlinear equations.

Hence we can summarize the results obtained in the following Theorem:

Theorem 1 *Apart from the class of equations linearizable by a Möbious transformation, which can be represented up to a Möbious transformation of the dependent variable (eventually composed with an exchange of the independent variables $n \leftrightarrow m$) by the equation*

$$w_{0,1} w_{1,1}\left(w_{1,0} + \beta w_{0,0}\right) + w_{0,0} w_{1,0}\left(\gamma w_{1,1} + \delta w_{0,1}\right) + \epsilon w_{0,0} w_{1,0} w_{0,1} w_{1,1} = 0, \quad \epsilon = 0, 1, \quad \beta, \gamma, \delta \neq 0, \tag{21}$$

linearizable by the inversion $\widetilde{u}_{0,0} = 1/w_{0,0}$ to the equation

$$\widetilde{u}_{0,0} + \beta \widetilde{u}_{1,0} + \gamma \widetilde{u}_{0,1} + \delta \widetilde{u}_{1,1} + \epsilon = 0, \tag{22}$$

the only other linearizable equations are up to a Möbious transformation of the dependent variable (eventually composed with an exchange of the independent variables $n \leftrightarrow m$), represented by the following six nonlinear equations

$$\begin{aligned} w_{0,0}w_{1,0}w_{0,1}w_{1,1} - 1 &= 0, &\text{(23a)}\\ w_{0,0} - w_{1,0}w_{0,1}w_{1,1} &= 0, &\text{(23b)}\\ w_{1,0} - w_{0,0}w_{0,1}w_{1,1} &= 0, &\text{(23c)}\\ w_{1,1} - w_{0,0}w_{1,0}w_{0,1} &= 0, &\text{(23d)}\\ w_{0,1}w_{1,1} - \theta w_{0,0}w_{1,0} &= 0, &\text{(23e)}\\ w_{0,0}w_{1,1} - \theta w_{1,0}w_{0,1} &= 0, &\text{(23f)}\end{aligned}$$

where $\theta \neq 0$ is an otherwise arbitrary complex parameter. They are linearizable by the transformation $\widetilde{u}_{0,0} = \log w_{0,0}$ to the equations

$$\begin{aligned} \widetilde{u}_{0,0} + \widetilde{u}_{1,0} + \widetilde{u}_{0,1} + \widetilde{u}_{1,1} &= 2\pi \mathrm{i} z, &\text{(23g)}\\ -\widetilde{u}_{0,0} + \widetilde{u}_{1,0} + \widetilde{u}_{0,1} + \widetilde{u}_{1,1} &= 2\pi \mathrm{i} z, &\text{(23h)}\\ \widetilde{u}_{0,0} - \widetilde{u}_{1,0} + \widetilde{u}_{0,1} + \widetilde{u}_{1,1} &= 2\pi \mathrm{i} z, &\text{(23i)}\\ -\widetilde{u}_{0,0} - \widetilde{u}_{1,0} - \widetilde{u}_{0,1} + \widetilde{u}_{1,1} &= 2\pi \mathrm{i} z, &\text{(23j)}\\ -\widetilde{u}_{0,0} - \widetilde{u}_{1,0} + \widetilde{u}_{0,1} + \widetilde{u}_{1,1} &= \log\theta + 2\pi \mathrm{i} z, &\text{(23k)}\\ \widetilde{u}_{0,0} - \widetilde{u}_{1,0} - \widetilde{u}_{0,1} + \widetilde{u}_{1,1} &= \log\theta + 2\pi \mathrm{i} z, &\text{(23l)}\end{aligned}$$

where log *always stands for the principal branch of the complex logarithmic function and where* $\log\theta$ *stands for the principal branch of the complex logarithmic function of the parameter θ.*

Let's note that, given a solution $w_{0,0}$ of eqs. (23a-23f), the choice of the principal branch of the complex logarithm function in the transformation $\widetilde{u}_{0,0} = \log w_{0,0}$ reflects in the inhomogeneous terms of the linear eqs. (23g-23l) and is always $|z| \leq 2$. This can be easily seen considering that we must have $2\pi|z| \leq |Im\,(u_{0,0})\,| + |Im\,(u_{1,0})\,| + |Im\,(u_{0,1})\,| + |Im\,(u_{1,1})\,| \leq 4\pi$ (with a slight difference in (23k) and (23l)). Through the translation $\widetilde{u}_{0,0} = v_{0,0} + \pi \mathrm{i} z/2$ the linear equation (23g) can be made homogeneous. As a consequence in (23a) $w_{0,0} = e^{v_{0,0}} e^{\pi \mathrm{i} z/2}$, so that the integer parameter z can take the values
$z = -1,\ 0,\ 1,\ 2$. Hence we conclude that the term $2\pi \mathrm{i} z$ in (23g) takes into account the discrete symmetry of (23a) given by $w_{0,0} \to \zeta w_{0,0}$, where ζ stands for one of the four roots of unity $\zeta = 1,\ -1,\ \mathrm{i}$ and $-\mathrm{i}$. The same happens when applying the translation $\widetilde{u}_{0,0} = v_{0,0} + \pi \mathrm{i} z$ to the linear equations (23h, 23i) and $\widetilde{u}_{0,0} = v_{0,0} - \pi \mathrm{i} z$ to (23j). In (23b, 23c, 23d) we have $w_{0,0} = e^{v_{0,0}} e^{\pm \pi \mathrm{i} z}$ so that, without any loss of generality, we can restrict the values of z to $z = 0,\ 1$. The term $2\pi \mathrm{i} z$ in (23h, 23i, 23j) takes into account the discrete symmetry of (23b, 23c, 23d) given by $w_{0,0} \to \zeta w_{0,0}$, where ζ stands for one of the two roots of unity $\zeta = 1$ and -1. The same happens for the linear equations (23k) when we consider the non autonomous translation $\widetilde{u}_{0,0} = v_{0,0} + (\log\theta + 2\pi \mathrm{i} z)\, m/2$. In (23e) $w_{0,0} = e^{v_{0,0}} e^{\pi \mathrm{i} m z} \theta^{m/2}$ so that, without any loss of generality, we can also restrict the values z to $z = 0,\ 1$. Hence we conclude that the term $2\pi \mathrm{i} z$ in (23k) takes into account the discrete symmetry of (23e) given by $w_{0,0} \to (-1)^m\, w_{0,0}$. The same happens for the linear equation (23l) with the non autonomous translation $\widetilde{u}_{0,0} = v_{0,0} + (\log\theta + 2\pi \mathrm{i} z)\, nm$. Now, as $e^{2\pi \mathrm{i} nmz} = 1$ for any z, in (23f) $w_{0,0} = e^{v_{0,0}} \theta^{nm}$ so that, without any loss of generality, we can choose $z = 0$.

4 Concluding remarks and outlook

In this paper we have classified all multilinear partial difference equations which can be linearized by a point transformation. The resulting linearizable equations are presented in Theorem 1 together with their linearized counterparts.

It seems interesting at this point to try to analyze the case of more complicate classes of equations involving more lattice points and which could provide in the continuous case parabolic or elliptic partial differential equations. Work on this is in progress.

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